

SPDH-Sign

Towards Efficient, Post-quantum, Group-based Signatures

Christopher Battarbee¹ Delaram Kahrobaei²
Ludovic Perret³ Siamak F. Shahandashti¹

¹University of York

²City University of New York

³Sorbonne University

August 7, 2023

What is SPDH-Sign?



- Semidirect Product Diffie-Hellman Signatures

¹Jean-Marc Couveignes. “Hard homogeneous spaces”. In: *Cryptology ePrint Archive* (2006), Alexander Rostovtsev and Anton Stolbunov. “Public-key cryptosystem based on isogenies”. In: *Cryptology ePrint Archive* (2006).

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- Semidirect Product Diffie-Hellman Signatures
- A Couveignes-Rostostev-Stolbunov¹ style signature scheme based on group actions arising from group-based cryptography (CRS schemes)

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What is SPDH-Sign?



- Semidirect Product Diffie-Hellman Signatures
- A Couveignes-Rostostev-Stolbunov¹ style signature scheme based on group actions arising from group-based cryptography (CRS schemes)
- Addresses a problem with efficient sampling found in similar schemes

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Group Actions



Definition

A group action (G, X, \circledast) consists of a finite abelian group G , a set X , and a function $\circledast : G \times X \rightarrow X$ such that for all $g, h \in G, x \in X$

- $(g + h) \circledast x = g \circledast (h \circledast x)$
- $0 \circledast x = x$

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We are interested in **free, transitive** group actions: *only* 0 fixes every element of x , every pair (x, y) has a (unique) element $g \in G$ such that $g \circledast x = y$.

A Sigma Protocol

$$X_0, X_1 \in X, s \circledast X_0 = X_1$$



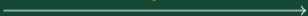
P

V

$$r \xleftarrow{\$} G$$

$$I \leftarrow r \circledast X_0$$

I



$$c \xleftarrow{\$} \{0, 1\}$$

c



$$p \leftarrow r - c \cdot s$$

p



Transcripts (I, c, p) are passing if

$$p \circledast X_c \stackrel{?}{=} I$$

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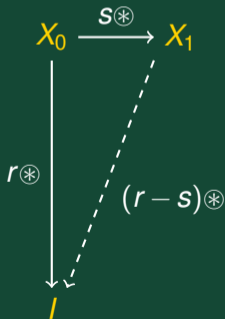
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Fiat-Shamir Transform



- Uses hash functions into the **challenge space** to generate sigma protocol transcripts non-interactively

²Ward Beullens et al. “CSI-FiSh: efficient isogeny based signatures through class group computations”. In: *ASIACRYPT. 2019*.

³Jelle Don et al. “Security of the Fiat-Shamir transformation in the quantum random-oracle model”. In: *CRYPTO2019*.

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- Gives 'secure' signature schemes provided hash functions are modelled as **random oracles**
- As shown (roughly) in² results of³ (**QR**OM) go through given **special soundness** and **honest verifier zero knowledge**.
- Provided large enough challenge space, security bounded by advantage against recovering s from X_0, X_1 ; so-called **group action discrete logarithm problem** admitting quantum subexponential algorithms

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Honest Verifier Zero Knowledge



Definition

Our sigma protocol has **HVZK** if given c we can efficiently produce a transcript (\bar{l}, c, \bar{p}) such that

- $\bar{p} \otimes X_c \stackrel{?}{=} \bar{p}$
- (\bar{l}, c, \bar{p}) has the same distribution as an honestly generated transcript

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- Given c choose $r \xleftarrow{\$} G$ and output $(r \otimes X_c, c, r)$.

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- Given c choose $r \xleftarrow{\$} G$ and output $(r \otimes X_c, c, r)$.
- Recall honest transcripts look like

$$(r \otimes X_0, 0, r) \text{ or } (r \otimes X_0, 1, r - s)$$

- In other words we have HVZK **provided we can sample uniformly at random**

Sampling



- Ability to sample uniformly gives zero knowledge property

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Sampling



- Ability to sample uniformly gives zero knowledge property
- Standard / original group action comes from isogenies; group hard to compute

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Sampling



- Ability to sample uniformly gives zero knowledge property
- Standard / original group action comes from isogenies; group hard to compute
- Workarounds include a one-time expensive calculation⁴ and ‘Fiat-Shamir with aborts’⁵

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Recap



- Certain types of group actions give short post-quantum signatures

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Recap



- Certain types of group actions give short post-quantum signatures
- Important to be able to compute the group of the group action or the security proofs fall apart
- Mainstream example of cryptographic group actions are such that it is difficult to compute the group
- Current solutions are slow to compute signatures or to generate parameters
- Would be nice to have a group action where we could compute the group easily...

The Semidirect Product



Definition

Let G be a finite group and $Aut(G)$ be its group of automorphisms. The semidirect product of G by $Aut(G)$ (written $G \rtimes Aut(G)$) is the group $G \times Aut(G)$ equipped with multiplication

$$(g, \phi)(h, \psi) = (\psi(g)h, \psi\phi)$$

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$$(g, \phi)(h, \psi) = (\psi(g)h, \psi\phi)$$

Definition

Let G be a finite group. Each pair $(g, \phi) \in G \rtimes Aut(G)$ defines a function $s_{g, \phi} : \mathbb{Z} \rightarrow G$ by

$$(g, \phi)^x = (s_{g, \phi}(x), \phi^x)$$

Acting by Integers



Notice⁶

$$\begin{aligned} (s_{g,\phi}(x+y), \phi^{x+y}) &= (g, \phi)^{x+y} \\ &= (g, \phi)^x (g, \phi)^y \\ &= (s_{g,\phi}(x), \phi^x) (s_{g,\phi}(y), \phi^y) \\ &= (\phi^y (s_{g,\phi}(x)) s_{g,\phi}(y), \phi^{x+y}) \end{aligned}$$

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There is a function $*$: $\mathbb{Z} \times G \rightarrow G$ such that

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$$y * s_{g,\phi}(x) = s_{g,\phi}(x+y)$$

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Looping

- Some $N \in \mathbb{N}$ is such that $(g, \phi)^N = (1, id)$, so $s_{g, \phi}(N) = 1$



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One has

$$\begin{aligned}(n + y) * s_{g, \phi}(x) &= y * s_{g, \phi}(x + n) \\ &= y * \phi^x(s_{g, \phi}(n))s_{g, \phi}(x) \\ &= y * s_{g, \phi}(x)\end{aligned}$$

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For efficient sampling: **how do we compute n ?**

Main Theorem



Theorem

$$s_{g,\phi}(x) = \phi^{x-1}(g)\dots\phi(g)g$$

Proof.

Induction - notice $s_{g,\phi}(x+1) = 1 * s_{g,\phi}(x) = \phi(s_{g,\phi}(x))g$



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Theorem

Let N be the order of (g, ϕ) as a $G \rtimes \text{Aut}(G)$ element, and n be the smallest integer such that $s_{g,\phi}(n) = 1$. Then $n|N$.

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Proof.

Certainly $N = kn + l$ for $k, l \in \mathbb{N}$, and $s_{g,\phi}(N) = 1$. We know

$$1 = \phi^{kn+l-1}(g)\dots\phi(g)g$$



Main Theorem

Proof.

We can write the set $\{0, \dots, kn + l - 1\}$ as

$$0 + \{0, \dots, n - 1\}$$

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...

$$(k - 1)n + \{0, \dots, n - 1\}$$

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In other words

$$\phi^{kn} \left(\phi^{l-1}(g) \dots g \right) \prod_{i=0}^{k-1} \phi^{(k-(i+1))n} \left(\phi^{n-1}(g) \dots g \right) = 1$$

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We therefore have $s_{g,\phi}(l) = 1$ with $l < n$, so $l = 0$.



Example



Let p be an odd prime, and define

$$G_p = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z}_{p^2}, a \equiv 1 \pmod{p} \right\}$$

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- 5 such values with additional restriction $n \leq p^3$
- Each such check requires $\mathcal{O}(\log p)$ semidirect product group operations

Parameter Estimation



- Complexity of quantum attacks as a function of size of group in group action

⁷Wouter Castryck et al. “CSIDH: an efficient post-quantum commutative group action”. In: *ASIACRYPT*. 2018, Xavier Bonnetain and André Schrottenloher. “Quantum security analysis of CSIDH”. In: *EUROCRYPT*. 2020.

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- Borrowing estimates from isogeny group action⁷ suggesting group size $\log n = 512$
- Tradeoffs available - at the short signature / long public key end signatures look like 8 pairs of (proof, challenge) with challenge space size 2^{16}
- Signatures of 538B at NIST 1 parameters

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~ The End ~