# NTWE: A Natural Combination of NTRU and LWE

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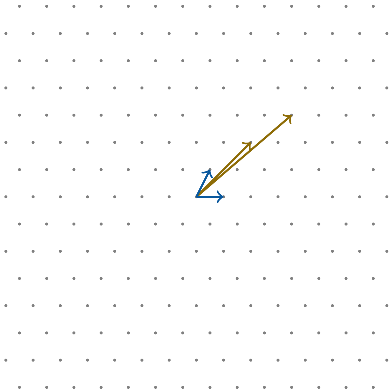
- Consider both the provable and concrete hardness of NTWE problem
- Construct and parametrize a NTWE-based cryptosystem

# Lattice-Based Cryptography

- Primary candidate for post-quantum cryptography
- Lattice-based KEM and signature algorithm to be standardized by NIST



# Assumptions in Lattice-Based Cryptography

- Primary two building blocks are NTRU [HPS98] and LWE [Reg05] problems
- Interesting to investigate alternative hardness assumptions
- We introduce the NTWE problem as a new problem for lattice-based cryptography

# Problem Statements

- Use ring $R_q = \mathbb{Z}_q[X]/(X^n + 1)$ with integers $n, q$
- Typical parameters $q = 3329$, $n = 256$ and $k < 5$

### Module-LWE $\mathrm{MLWE}(k)$

Distinguish between uniformly random $(\overline{\boldsymbol{A}} \in R_q^{m \times k}, \overline{\boldsymbol{b}} \in R_q^m)$ and $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}} = \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{s}} + \overline{\boldsymbol{e}})$ with uniformly random $\overline{\boldsymbol{A}}$ and small $\overline{\boldsymbol{s}} \in R_q^k$, $\overline{\boldsymbol{e}} \in R_q^m$

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### NTRU

Distinguish between uniformly random $h \in R_q$ and $h = gf^{-1}$ for small $g, f \in R_q$

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**Module NTRU $\mathrm{MNTRU}(k)$**

Distinguish between uniformly random $\overline{\boldsymbol{H}} \in R_q^{m \times k}$ and $\overline{\boldsymbol{H}} = \overline{\boldsymbol{G}}\overline{\boldsymbol{F}}^{-1}$ and small $\overline{\boldsymbol{G}} \in R_q^{m \times k}, \overline{\boldsymbol{F}} \in R_q^{k \times k}$

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### NTWE Problem $\mathrm{NTWE}(k)$

Distinguish between uniformly random $(\overline{\boldsymbol{A}} \in R_q^{m \times k}, \overline{\boldsymbol{b}} \in R_q^m)$ and $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}} = (\overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{s}} + \overline{\boldsymbol{e}})f^{-1})$ with uniformly random $\overline{\boldsymbol{A}}$, small $\overline{\boldsymbol{e}} \in R_q^m$, $\overline{\boldsymbol{s}} \in R_q^k$ and $f \in R_q$

# Problem Hardness

## Provable Hardness

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- Can easily see that the NTWE problem is not easier than either of these problems

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## Concrete Hardness

- NTWE problem naturally corresponds to a lattice problem
- The hardness of this lattice problem gives a concrete hardness estimate for the NTWE problem

# Provable Relation to MLWE problem

- Given $\mathrm{MLWE}(k)$ instance $\overline{\boldsymbol{A}} \in \mathbb{R}_q^{m \times k}, \overline{\boldsymbol{b}} \in \mathbb{R}_q^m$
- Sample $f$ from correct distribution and $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}} \cdot f^{-1})$ is an $\mathrm{NTWE}(k)$ instance
- Solving $\mathrm{NTWE}(k)$ instance gives solution to original $\mathrm{MLWE}(k)$ instance

# Provable Relation to NTRU Problem

- Given NTRU instance with multiple samples $h_i = g_i f^{-1}$
- Sample $\boldsymbol{A}$ uniformly at random and produce NTWE instance with

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{h}_{[1,\dots,k]} + \boldsymbol{h}_{[k+1,\dots k+m]} \left(\boldsymbol{A}\boldsymbol{g}_{[1,\dots,k]} + \boldsymbol{g}_{[k+1,\dots k+m]}\right) \cdot f^{-1}$$

- Solving NTWE instance implies solution to original NTRU instance

# Reduction From More Structured NTWE to MNTRU Problem

- More structured variant of $\mathrm{NTWE}(k)$ is at least as hard as $\mathrm{MNTRU}(k+1)$
- Additional structure might make the NTWE problem harder
- More natural to assume that less structure corresponds to harder problems

# Concrete Hardness of NTWE Problem

- NTWE corresponds to problem of finding unusually short vector in a lattice

**NTWE($k$) Lattice**

Lattice given by $\boldsymbol{A} \in \mathbb{Z}_q^{mn \times kn}$, $\boldsymbol{B} \in \mathbb{Z}_q^{mn \times n}$

# Concrete Hardness of NTWE Problem

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- $\mathrm{NTWE}(k)$ with $\boldsymbol{H} = \boldsymbol{A} \parallel \boldsymbol{B}$ gives same type of lattice as $\mathrm{MNTRU}(k+1)$

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**$\mathrm{MNTRU}(k+1)$ Lattice**

Lattice given by $\boldsymbol{H} \in \mathbb{Z}_q^{mn \times (k+1)n}$

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**$\mathrm{MNTRU}(k+1)$ Lattice**

Lattice given by $\boldsymbol{H} \in \mathbb{Z}_q^{mn \times (k+1)n}$

**$\mathrm{MLWE}(k+1)$ Lattice**

Lattice given by $\boldsymbol{A} \in \mathbb{Z}_q^{mn \times (k+1)n}$, $\boldsymbol{b} \in \mathbb{Z}_q^{mn \times 1}$

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- $\mathrm{NTWE}(k)$ lattice very similar to $\mathrm{MLWE}(k+1)$ lattice
- Concrete hardness of these types of lattice problems is well studied [CN11, Che13, APS15, ACD$^+$18]

**$\mathrm{NTWE}(k)$ Lattice**

Lattice given by $\boldsymbol{A} \in \mathbb{Z}_q^{mn \times kn}$, $\boldsymbol{B} \in \mathbb{Z}_q^{mn \times n}$

**$\mathrm{MNTRU}(k+1)$ Lattice**

Lattice given by $\boldsymbol{H} \in \mathbb{Z}_q^{mn \times (k+1)n}$

**$\mathrm{MLWE}(k+1)$ Lattice**

Lattice given by $\boldsymbol{A} \in \mathbb{Z}_q^{mn \times (k+1)n}$, $\boldsymbol{b} \in \mathbb{Z}_q^{mn \times 1}$

# NTWE-Based Cryptosystem

Alice → Bob: $(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}} = (\overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{s}} + \overline{\boldsymbol{e}})f^{-1})$

Bob → Alice: $c_1 = \overline{\boldsymbol{s}}' \cdot \overline{\boldsymbol{b}} + e' + \lfloor Mq/2 \rceil$, $\overline{\boldsymbol{c}}_2 = \overline{\boldsymbol{s}}' \cdot \overline{\boldsymbol{A}} + \overline{\boldsymbol{e}}''$

$$M = \lfloor 2(c_1 f - \overline{\boldsymbol{c}}_2 \cdot \overline{\boldsymbol{s}})/q \rceil \cdot f_2^{-1}$$

## Cryptosystem Construction

- Use public matrix $\overline{\boldsymbol{A}} \in R_q^{k \times (k-1)}$
- Ciphertext is $k$ samples from $\mathrm{MLWE}(k)$ instance
- Decryption correct as products of $\overline{\boldsymbol{s}}, \overline{\boldsymbol{e}}, f, \overline{\boldsymbol{s}}', e', \overline{\boldsymbol{e}}''$ small

# Comparable MLWE-Based Cryptosystem

Alice | Bob

$(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}} = \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{s}} + \overline{\boldsymbol{e}})$ $\longrightarrow$

$\longleftarrow$ $c_1 = \overline{\boldsymbol{s}}' \cdot \overline{\boldsymbol{b}} + e' + \lfloor Mq/2 \rceil$

$\overline{\boldsymbol{c}}_2 = \overline{\boldsymbol{s}}' \cdot \overline{\boldsymbol{A}} + \overline{\boldsymbol{e}}''$

$$M = \lfloor 2(c_1 - \overline{\boldsymbol{c}}_2 \cdot \overline{\boldsymbol{s}})/q \rceil$$

## Cryptosystem Construction

- Public matrix $\overline{\boldsymbol{A}} \in R_q^{k \times k}$
- Ciphertext is $k+1$ samples from $\mathrm{MLWE}(k)$ instance
- Decryption correct as products of $\overline{\boldsymbol{s}}, \overline{\boldsymbol{e}}, \overline{\boldsymbol{s}}', \overline{\boldsymbol{e}}''$ and $e'$ small

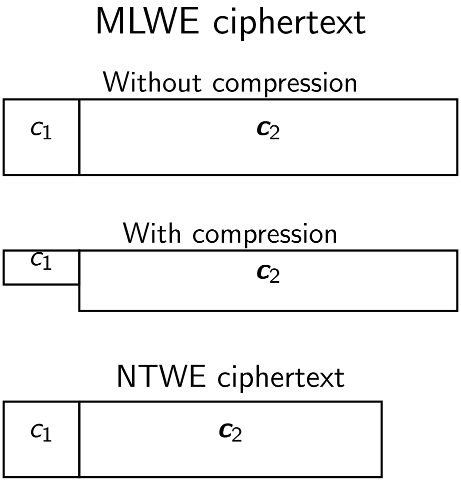
# Parametrizations of NTWE-Based Cryptosystem

- Parametrizations with same ring $R = \mathbb{Z}[X]/(X^{256} + 1)$ and modulos $q = 3329$ as in Kyber [SAB+22]
- Similar distributions for $\overline{\boldsymbol{s}}$ and $\overline{\boldsymbol{e}}$ as in Kyber
- More efficient than Kyber as requires fewer operations of equivalent cost

| Version | NTWE-768 | Kyber-768 | NTWE-1024 | Kyber-1024 |
|---|---|---|---|---|
| Core SVP | 182 | 183 | 256 | 256 |
| Dimension of $\overline{\boldsymbol{A}}$ | $3 \times 2$ | $3 \times 3$ | $4 \times 3$ | $4 \times 4$ |
| PK size (bytes) | 1184 | 1184 | 1568 | 1568 |
| CT size (bytes) | 1152 | 1088 | 1536 | 1568 |
| $\delta$ | $2^{-182}$ | $2^{-164}$ | $2^{-153}$ | $2^{-174}$ |

# Ciphertext Compression

- MLWE-based cryptosystems use ciphertext compression to allow for smaller ciphertexts
- Ciphertext compression not suitable for all applications
- NTWE-based cryptosystem without ciphertext compression allows more efficient encryption and decryption



# Efficiency of Operations

- Multiplication, inversion and addition in $R_q$ efficient with NTT
- Multiplication and inversion in $R_2$ during decryption
  less efficient as not performed with
- Sampling $f$ such that it is trivial in $R_2$ ensures no operations in $R_2$ are required

Alice | Bob

$$(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}} = (\overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{s}} + \overline{\boldsymbol{e}}) f^{-1}) \longrightarrow$$

$$(\overline{\boldsymbol{A}}, \overline{\boldsymbol{b}} = \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{s}} + \overline{\boldsymbol{e}})$$

$$\longleftarrow c_1 = \overline{\boldsymbol{s}}' \cdot \overline{\boldsymbol{b}} + e' + \lfloor Mq/2 \rceil$$

$$\overline{\boldsymbol{c}}_2 = \overline{\boldsymbol{s}}' \cdot \overline{\boldsymbol{A}} + \overline{\boldsymbol{e}}''$$

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Alice — Bob

$$\text{Alice} \xrightarrow[(\overline{\boldsymbol{A}},\, \overline{\boldsymbol{b}} = \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{s}} + \overline{\boldsymbol{e}})]{(\overline{\boldsymbol{A}},\, \overline{\boldsymbol{b}} = (\overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{s}} + \overline{\boldsymbol{e}}) f^{-1})} \text{Bob}$$

$$\text{Alice} \xleftarrow[\overline{\boldsymbol{c}}_2 = \overline{\boldsymbol{s}}' \cdot \overline{\boldsymbol{A}} + \overline{\boldsymbol{e}}'']{c_1 = \overline{\boldsymbol{s}}' \cdot \overline{\boldsymbol{b}} + e' + \lfloor Mq/2 \rceil} \text{Bob}$$

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# Conclusion

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- Estimated the concrete hardness of the NTWE problem based on the hardness of the natural corresponding lattice problem
- Constructed a NTWE-based cryptosystem with performance competitive with highly efficient lattice-based cryptosystems

Questions?

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