Isogeny-based cryptography
After the Snap

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Isogeny crypto

Elliptic curves, isogenies, computational problems

Elliptic curves

Elliptic curve over $\mathbb{F}_q$: solutions (x,y) in $\mathbb{F}_q$ of

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- $\deg(\varphi \circ \psi) = \deg(\varphi) \cdot \deg(\psi)$

The isogeny problem

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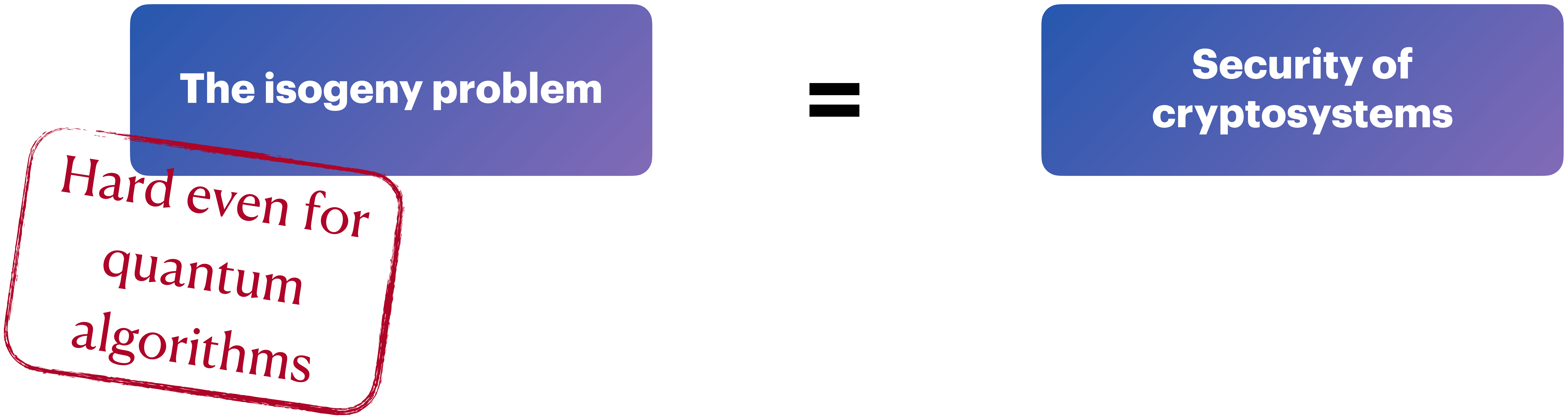

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Post-quantum cryptography

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- The solution φ is an isogeny…
- How to represent an isogeny?
- *Hint*: $\ker(\varphi)$ determines φ…

Efficient isogenies

- Given ker(φ) (list of points), can evaluate φ in poly time — Vélu's formulae
  - ✓ Isogenies of *small* degree ℓ = 2, or 3... "ℓ-isogenies"

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- Compose small isogenies to build bigger ones!

  ✓ Isogenies with **smooth degree** (small prime factors):

  $\varphi_n \circ \ldots \circ \varphi_2 \circ \varphi_1$ represented by ('compose', $\varphi_1, \varphi_2, \ldots, \varphi_n$), with $\deg(\varphi_i)$ small

Isogeny graph

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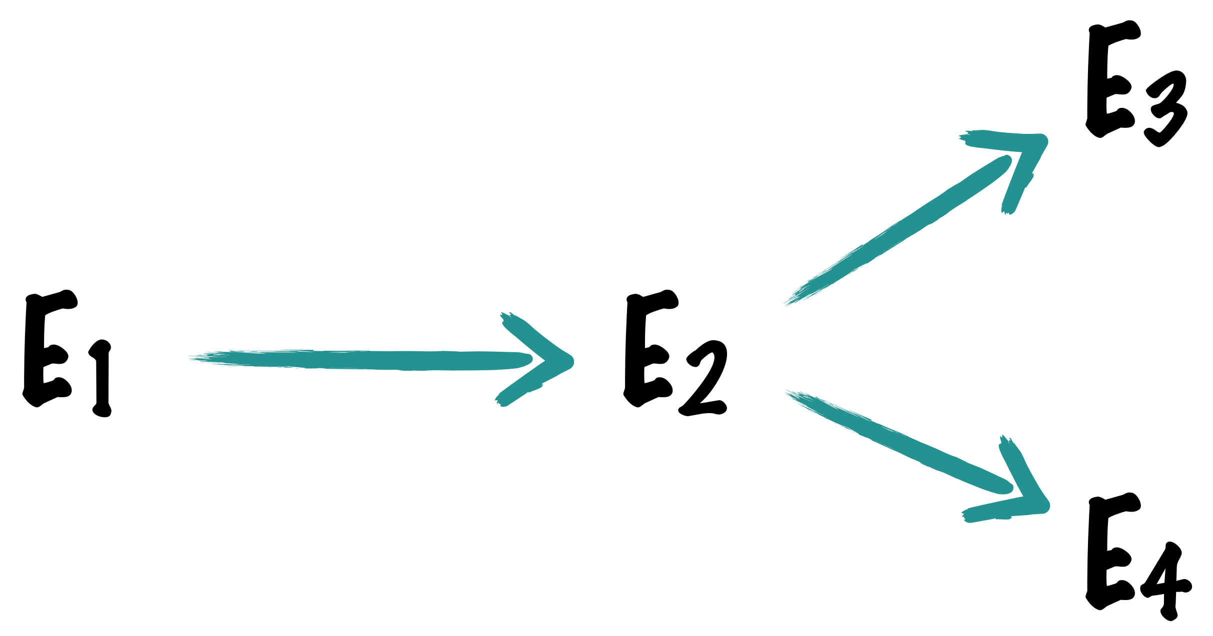

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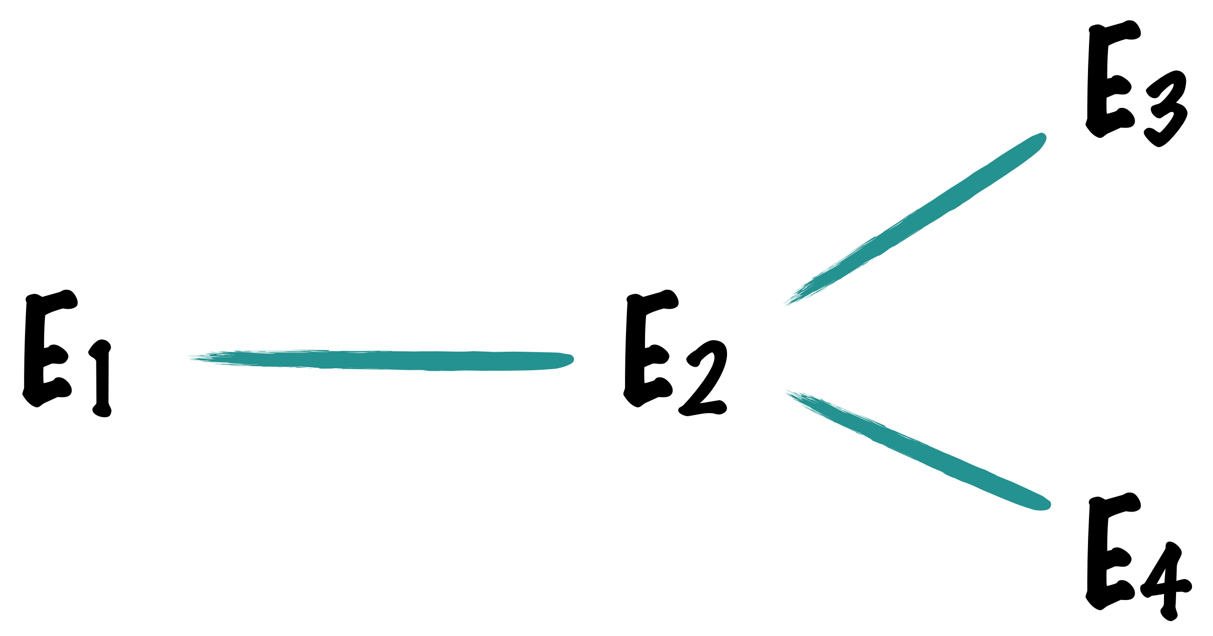

an isogeny of degree ℓ = an edge in a graph

$\exists\ \ell$-isogeny $E_1 \to E_2 \Rightarrow \exists\ \ell$-isogeny $E_2 \to E_1$

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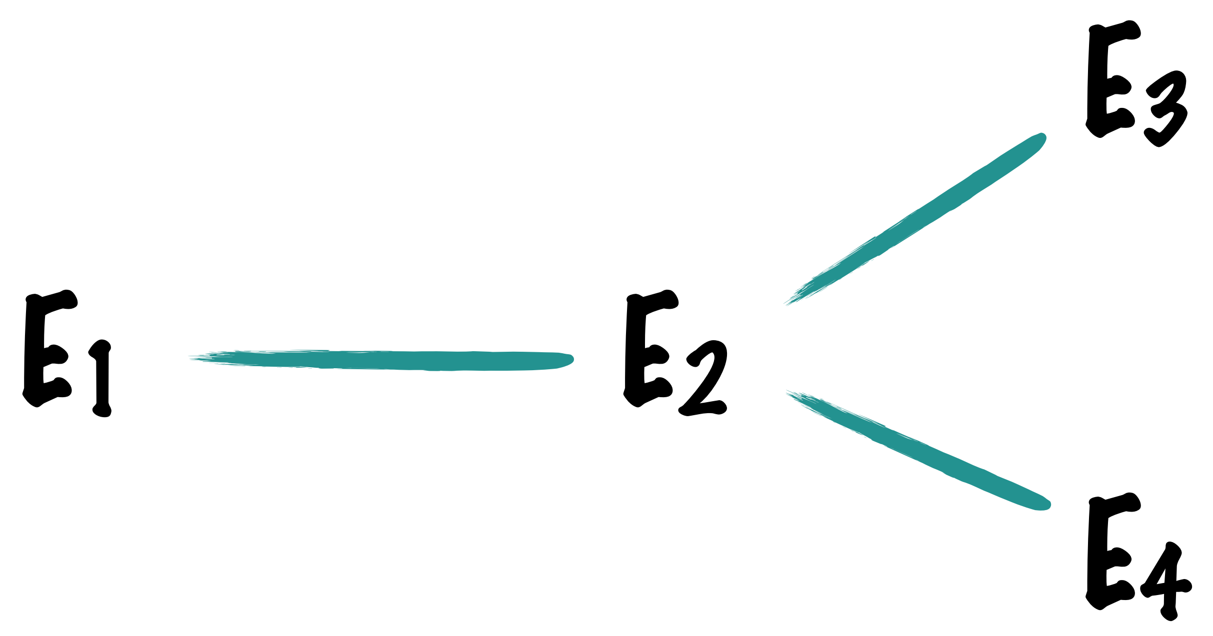
- **The ℓ-isogeny graph** (supersingular…)



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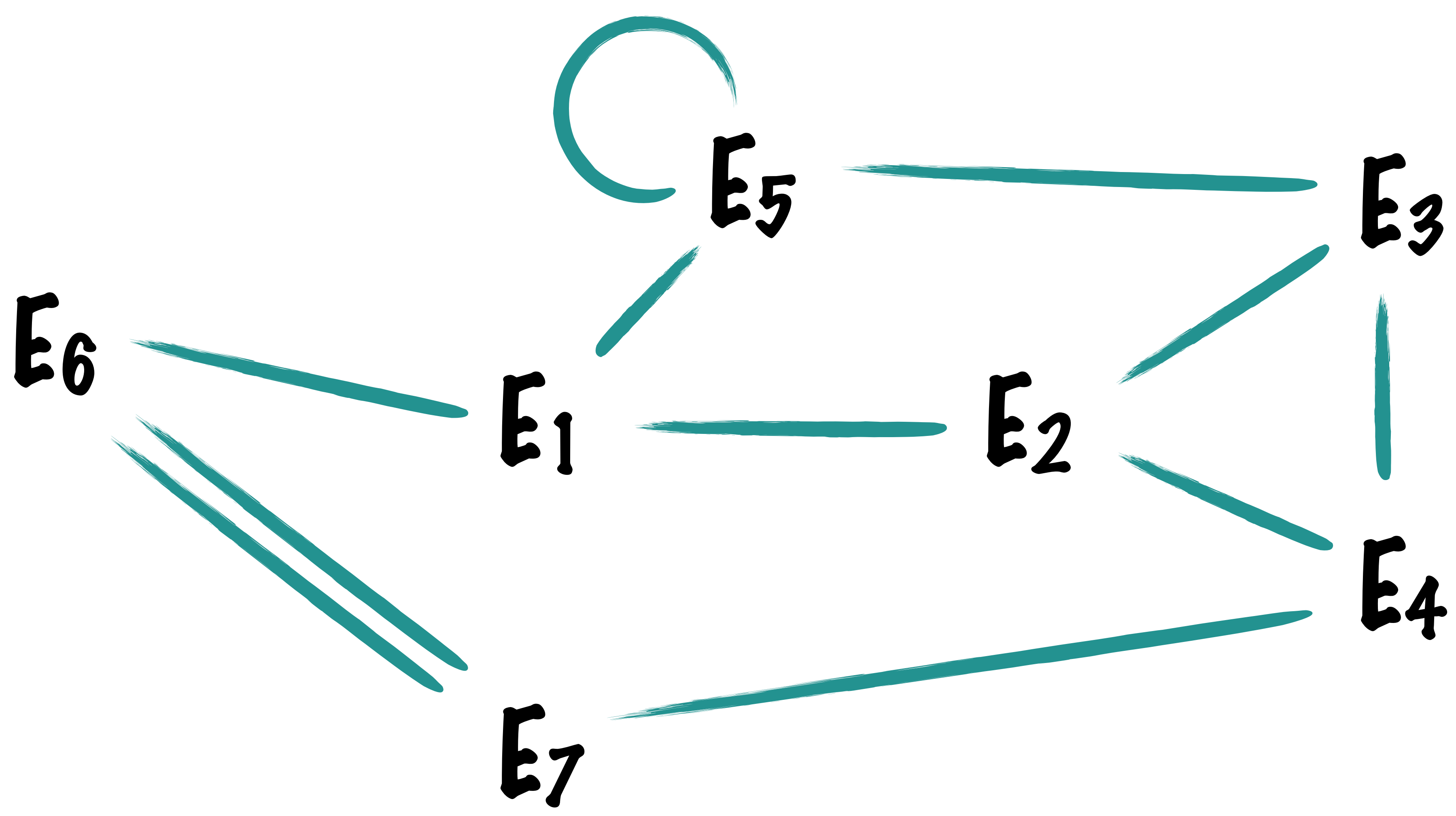

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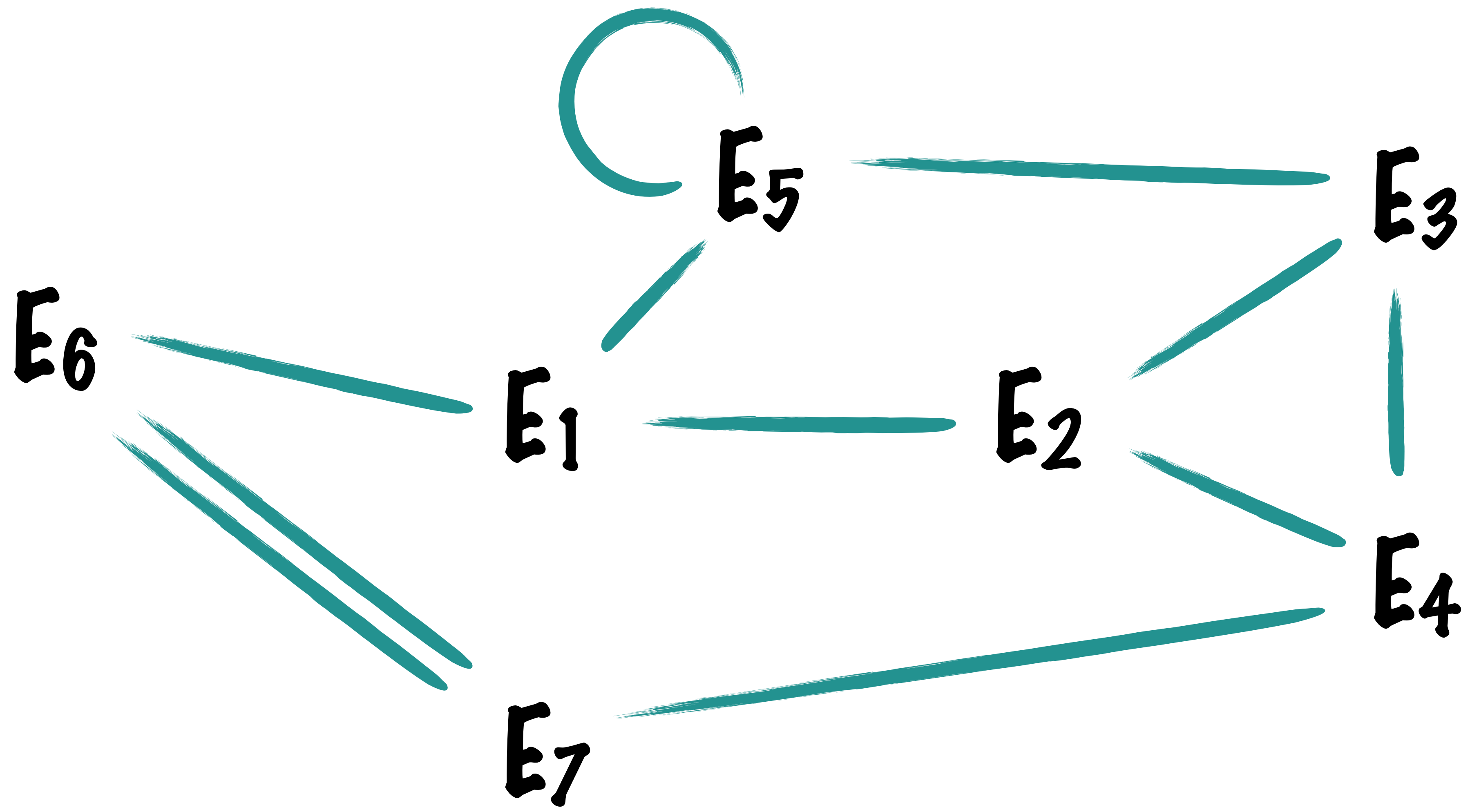

- $(\ell + 1)$-regular, **connected** (for supersingular curves)

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- Path finding in a graph
- Hard! Best known algorithms = generic graph algorithms
- Typical meaning of ***"the isogeny problem"***

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Reality: a mess

Weird scheme-dependent variants of isogeny problems $\leq$ **Security of cryptosystems** $\leq$ **The isogeny problem**

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One endomorphism	=	SQISign (soundness)

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[Jao, De Feo] PQCrypto 2011
Isogeny-based key exchange
NIST PQC alt-finalist

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The isogeny problem with "torsion point information"...

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SIDH

Jao–De Feo 2011

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  with $\ker(\varphi) = G$

- $\deg(\varphi) = \#G$
- Given generators of G, if $\#G$ has **only small prime factors**, then φ can be **computed efficiently**

SIDH

Fix reference elliptic curve E_0

Alice

Bob

SIDH

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Random subgroup G of E_0

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Random subgroup G of E_0

Compute $\varphi_A : E_0 \to E_0/G$

SIDH

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Compute $\varphi_A : E_0 \to E_0/G$

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SIDH

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$\xrightarrow{E_A}$

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Random subgroup H of E_0

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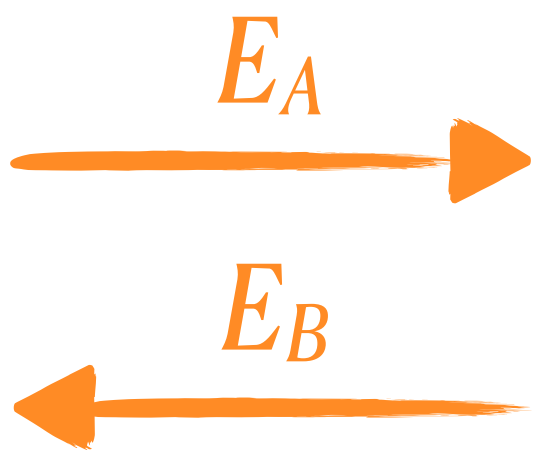

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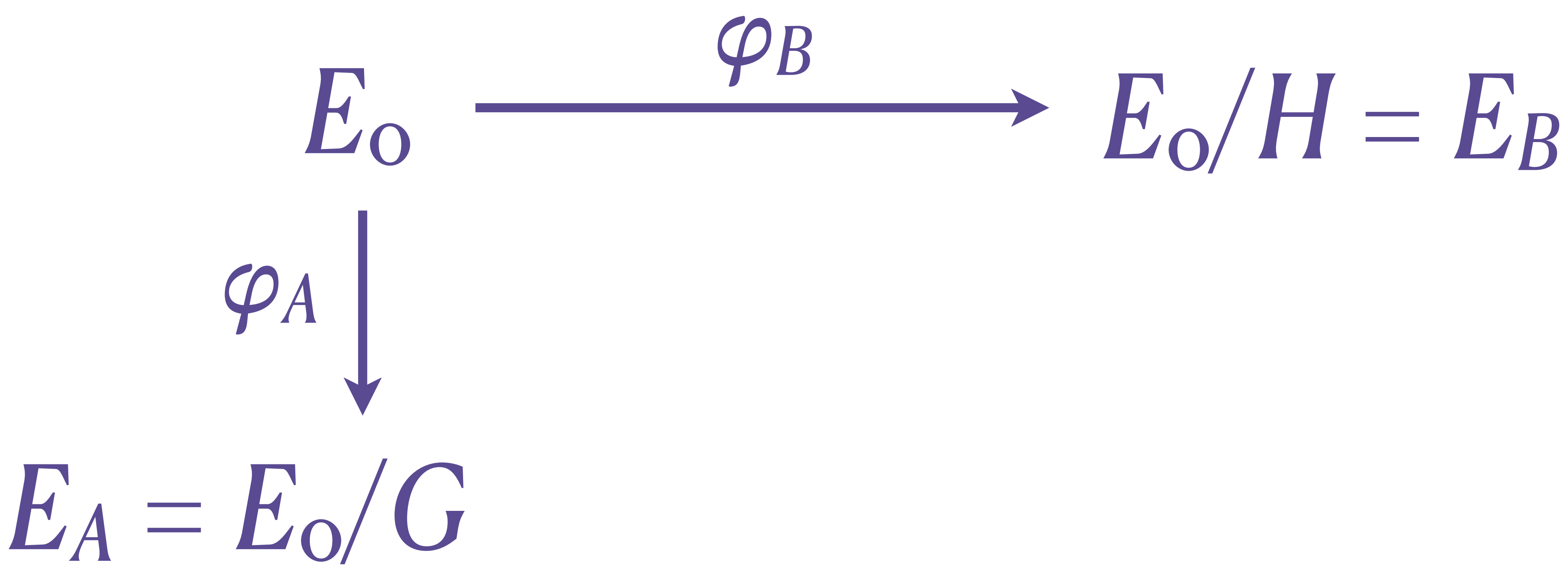

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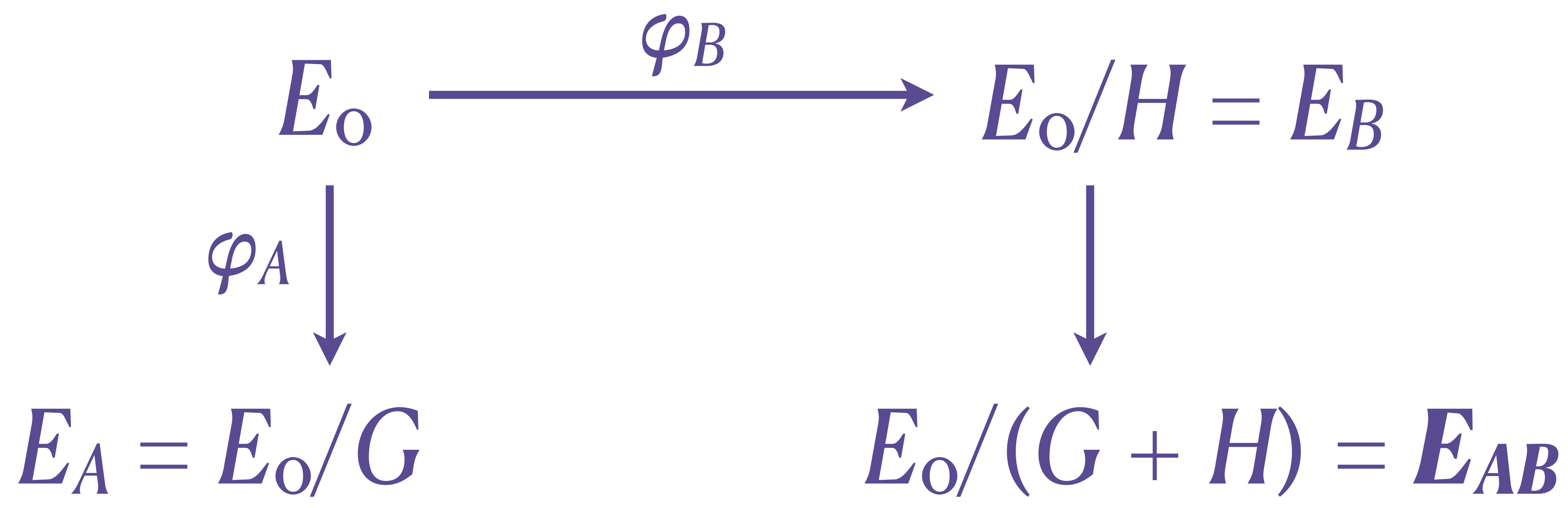

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Compute $\boldsymbol{E_{AB}} = E_B/G$

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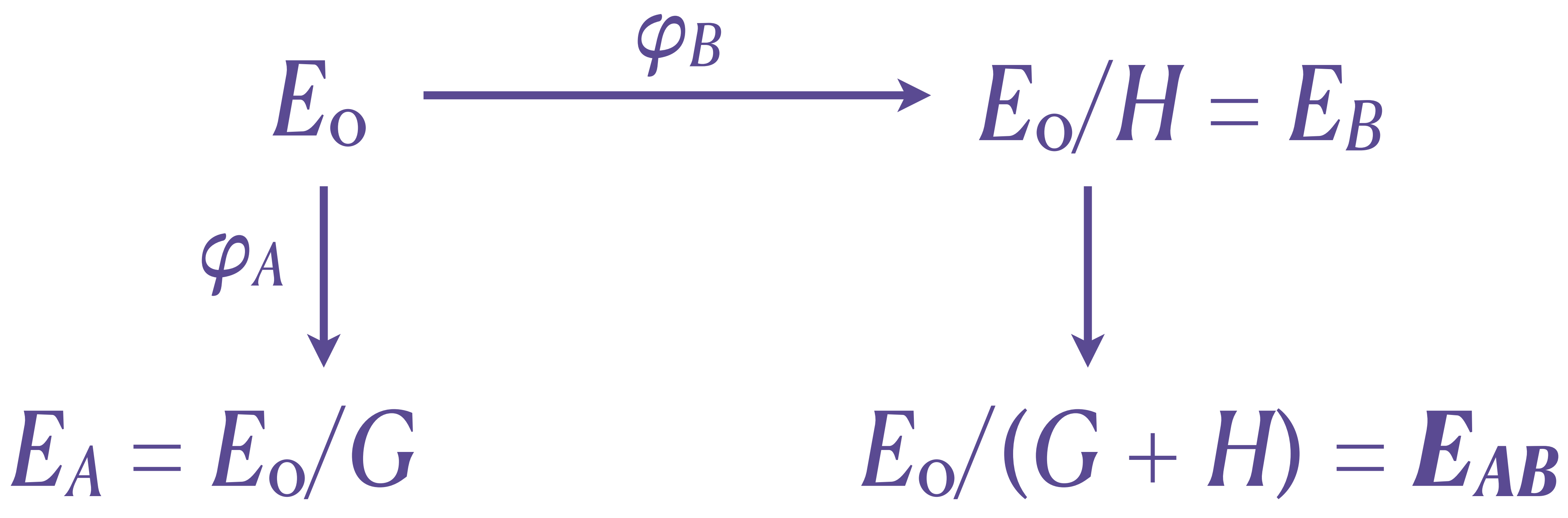

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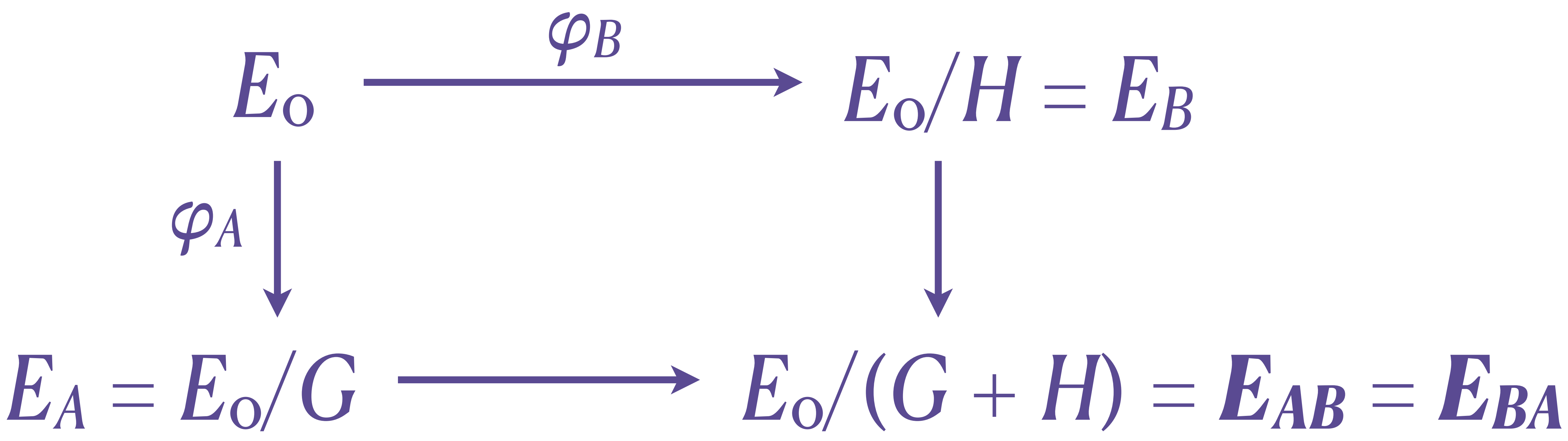

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Alice → Bob: E_A

Bob → Alice: E_B

G is not a subgroup of E_B

$\varphi_B(G)$ is!

SIDH

Fix reference elliptic curve E_0

Alice		**Bob**
Random subgroup G of E_0		Random subgroup H of E_0
Compute $\varphi_A : E_0 \to E_0/G$	E_A →	Compute $\varphi_B : E_0 \to E_0/H$
Let $E_A = E_0/G$	← E_B	Let $E_B = E_0/H$
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How to compute $\varphi_B(G)$?
Alice does not know φ_B...

Torsion

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- Alice can compute $\varphi_B(G)$

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- Bob gives φ_B on $E_0[2^n]$ ← φ_B remains secret everywhere else...
- Alice can compute $\varphi_B(G)$ ← Can compute shared secret $\boldsymbol{E_{AB}} = E_B/\varphi_B(G)$

SIDH

Fix: an elliptic curve E_0

Generators P_2, Q_2 of $E_0[2^n] \cong (\mathbb{Z}/2^n\mathbb{Z})^2$

Generators P_3, Q_3 of $E_0[3^m] \cong (\mathbb{Z}/3^m\mathbb{Z})^2$

Alice

Random subgroup G of $E_0[2^n]$

Compute $\varphi_A : E_0 \to E_0/G$

Let $E_A = E_0/G$

$E_A, \varphi_A(P_3), \varphi_A(Q_3)$ →

← $E_B, \varphi_B(P_2), \varphi_B(Q_2)$

Compute $\boldsymbol{E_{AB}} = E_B/\varphi_B(G)$

Bob

Random subgroup H of $E_0[3^m]$

Compute $\varphi_B : E_0 \to E_0/H$

Let $E_B = E_0/H$

Compute $\boldsymbol{E_{BA}} = E_A/\varphi_A(H)$

The SSI-T problem

Context:

- two elliptic curves E_0 and E_1
- an isogeny $\varphi : E_0 \to E_1$ (say, of degree 3^m like Bob's isogeny)
- an integer N coprime to $\deg(\varphi)$ (say, $N = 2^n$...)
- generators P and Q of $E_0[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$

SSI-T: Given E_0, E_1, P, Q, $\varphi(P)$ and $\varphi(Q)$, find the isogeny $\varphi : E_0 \to E_1$

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"torsion point information"

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SIDH key recovery ⇔ SSI-T

Torsion point information: a weakness?

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2011

Birth of SIDH

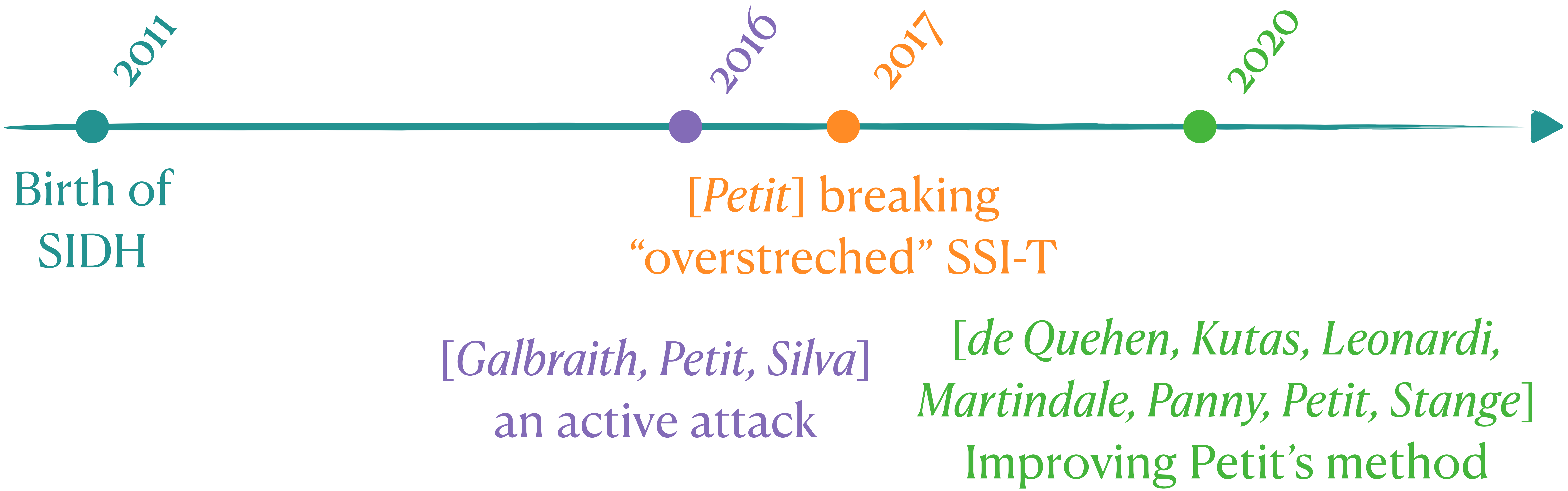
2016

[*Galbraith, Petit, Silva*] an active attack

2017

[*Petit*] breaking "overstreched" SSI-T

Torsion point information: a weakness?



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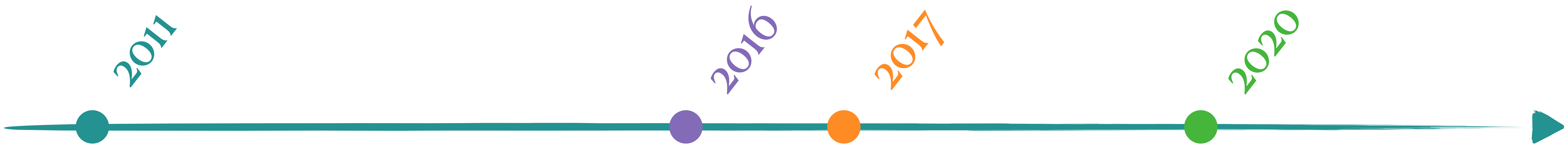

Birth of SIDH

[*Petit*] breaking "overstreched" SSI-T

[*Galbraith, Petit, Silva*] an active attack

[*de Quehen, Kutas, Leonardi, Martindale, Panny, Petit, Stange*] Improving Petit's method

Standard SIDH parameters totally unaffected

The Snap

July 30 2022


July 29 2022
Enjoying the French Alps

July 30 2022

eprint 2022/975

July 30 2022

eprint 2022/975

An efficient key recovery attack on SIDH

July 30 2022

eprint 2022/975

An efficient key recovery attack on SIDH

Wouter Castryck, Thomas Decru

July 30 2022

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An efficient key recovery attack on SIDH

Wouter Castryck, Thomas Decru

"Breaks SIKEp434 challenge in ten minutes"

Eurocrypt 2023 – “Isogeny 1” session

Efficient Key Recovery Attack on SIDH (Best Paper Award)

[Castryck, Decru]

A Direct Key Recovery Attack on SIDH (Honourable Mention)

[Maino, Martindale, Panny, Pope, W.]

Breaking SIDH in Polynomial Time (Honourable Mention)

[Robert]

Main result of the attacks

Interpolating isogenies [CD, MMPPW, R]**:**

- Let $\varphi : E_1 \to E_2$ of degree d
- Let $n > (\log_2(d) + 1)/2$, and (P, Q) is a basis of $E_1[2^n]$
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Corollary: The few points leaked by SIDH leak the full secret.

Isogeny-based cryptography

Body count

Weird scheme-dependent variants of isogeny problems	$\leq$	**Security of cryptosystems**	$\leq$	**The isogeny problem**

The isogeny problem	=	CGL hash function (preimage)
One endomorphism	=	SQISign (soundness)
Vectorisation	=	CSIDH (key recovery)
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~~SHealS~~

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  - ➡ Key exchange very similar to Diffie–Hellman
- Wide variety of *CSIDH-inspired* constructions
  - ➡ "group action" cryptography
  - ➡ Signatures, PRFs, threshold stuff, oblivious stuff…

Fixing SIDH?

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- Huge cost: 4434 bytes public keys (vs. 197 bytes in SIKE)

Representing isogenies

Back to the foundations

The isogeny problem

"Idealised" isogeny problem: Given E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$

ℓ-isogeny path problem: Given E_1 and E_2, find an ℓ-isogeny path from E_1 to E_2

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Examples:

- Small degree isogenies
- Compositions of small degree isogenies
- Linear combinations of compositions of small degree isogenies…

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- **"Interpolation representation"** of φ, or "HD representation"
- Universal! Given any efficient repr. of φ, can compute its interpolation repr.

The universal isogeny problem

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Universal isogeny $\Leftrightarrow$ ℓ-isogeny path

[Page, W.] to appear

From attacks to constructions

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How efficient is it?

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The attack

Isogenies in higher dimension

Dual

Let E an elliptic curve over $\mathbb{F}_q$ and N an integer

- Multiplication by N is an isogeny

$$[N] : E \to E : P \longmapsto [N]P = P + P + \ldots + P$$

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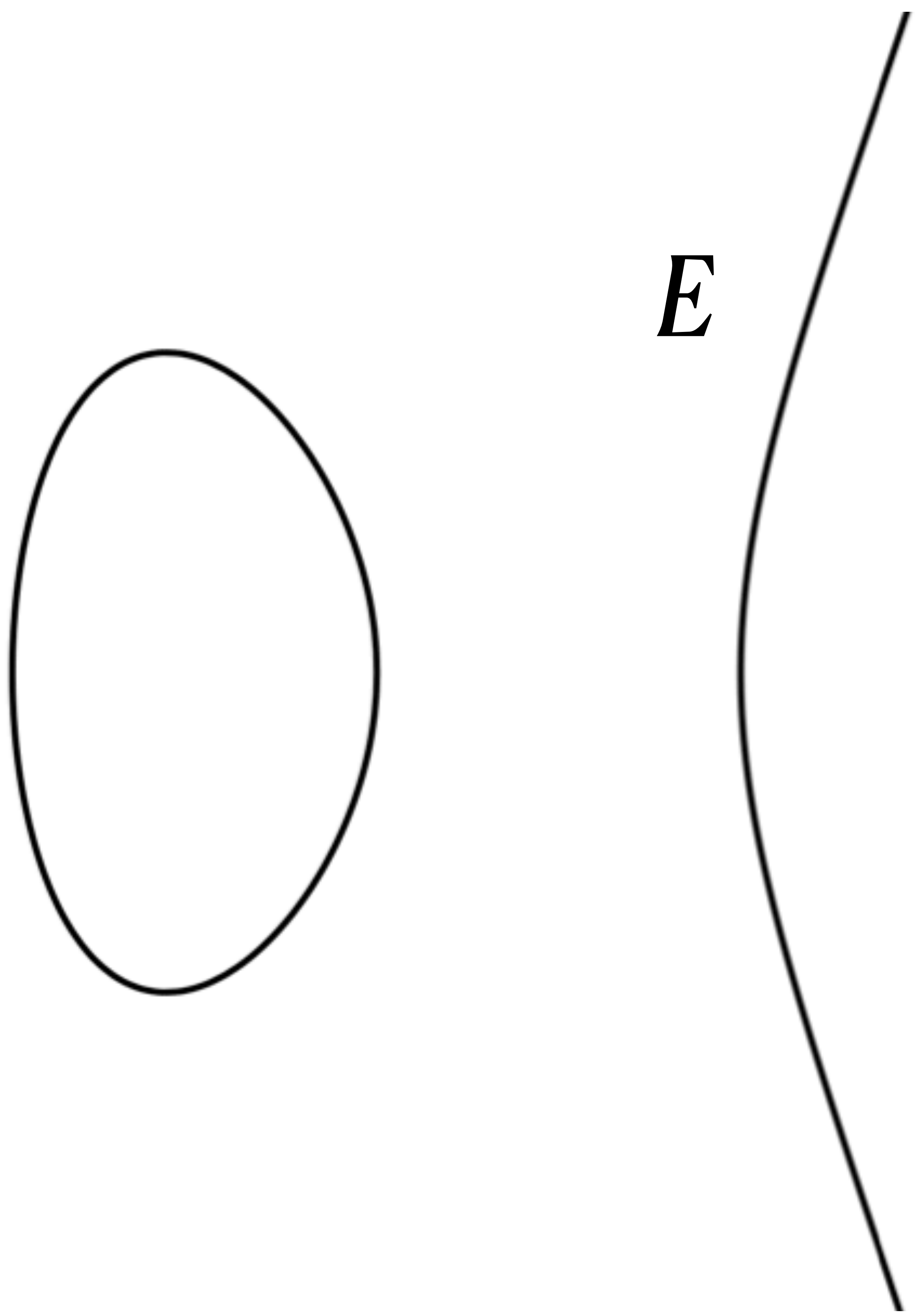
- Let $\varphi : E_1 \to E_2$ be an isogeny
- **Dual of φ:** unique isogeny $\hat{\varphi} : E_2 \to E_1$ such that

$$\hat{\varphi} \circ \varphi = [\deg(\varphi)]$$

Abelian varieties

Elliptic curve: a curve that is also a group



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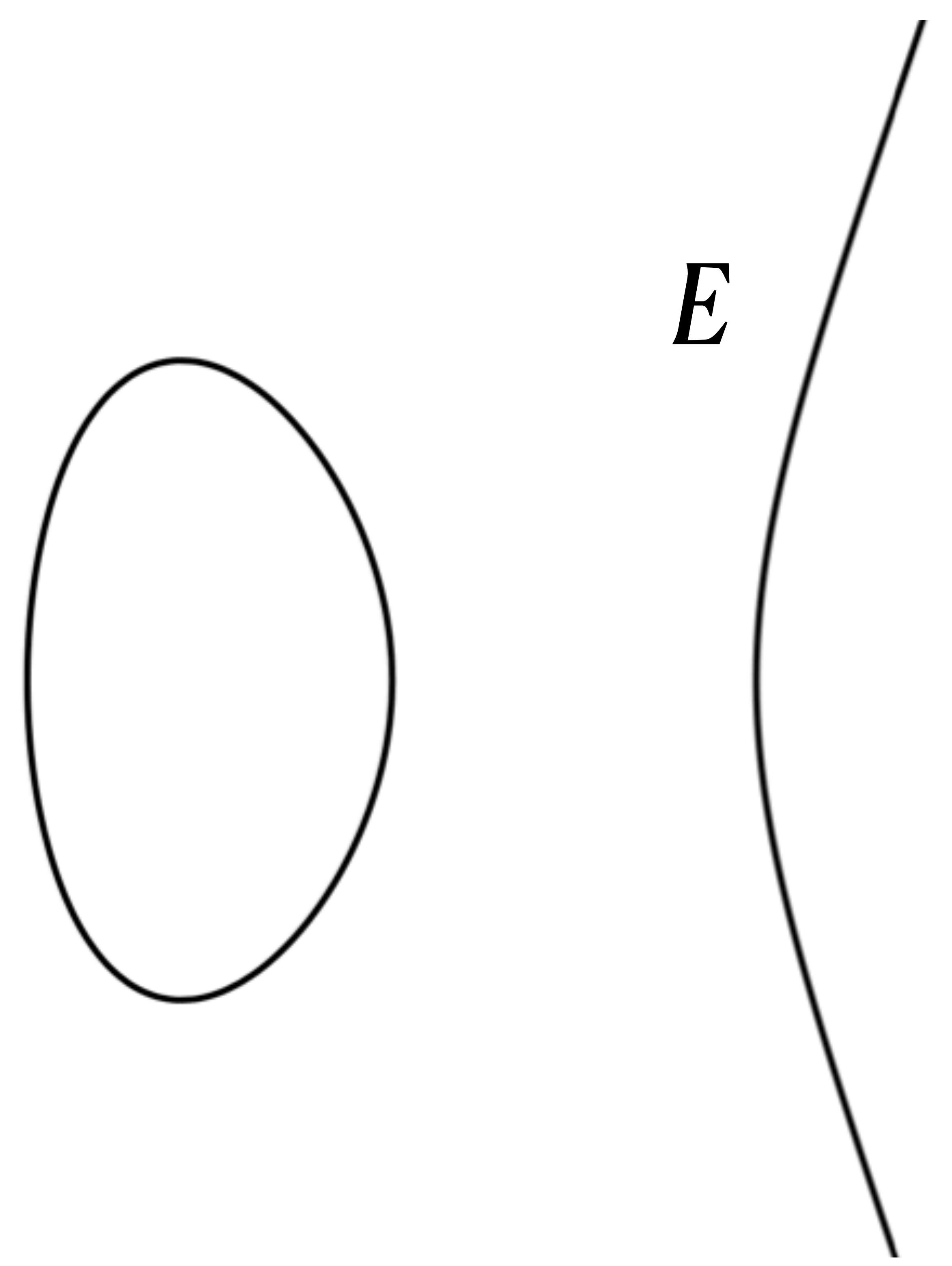
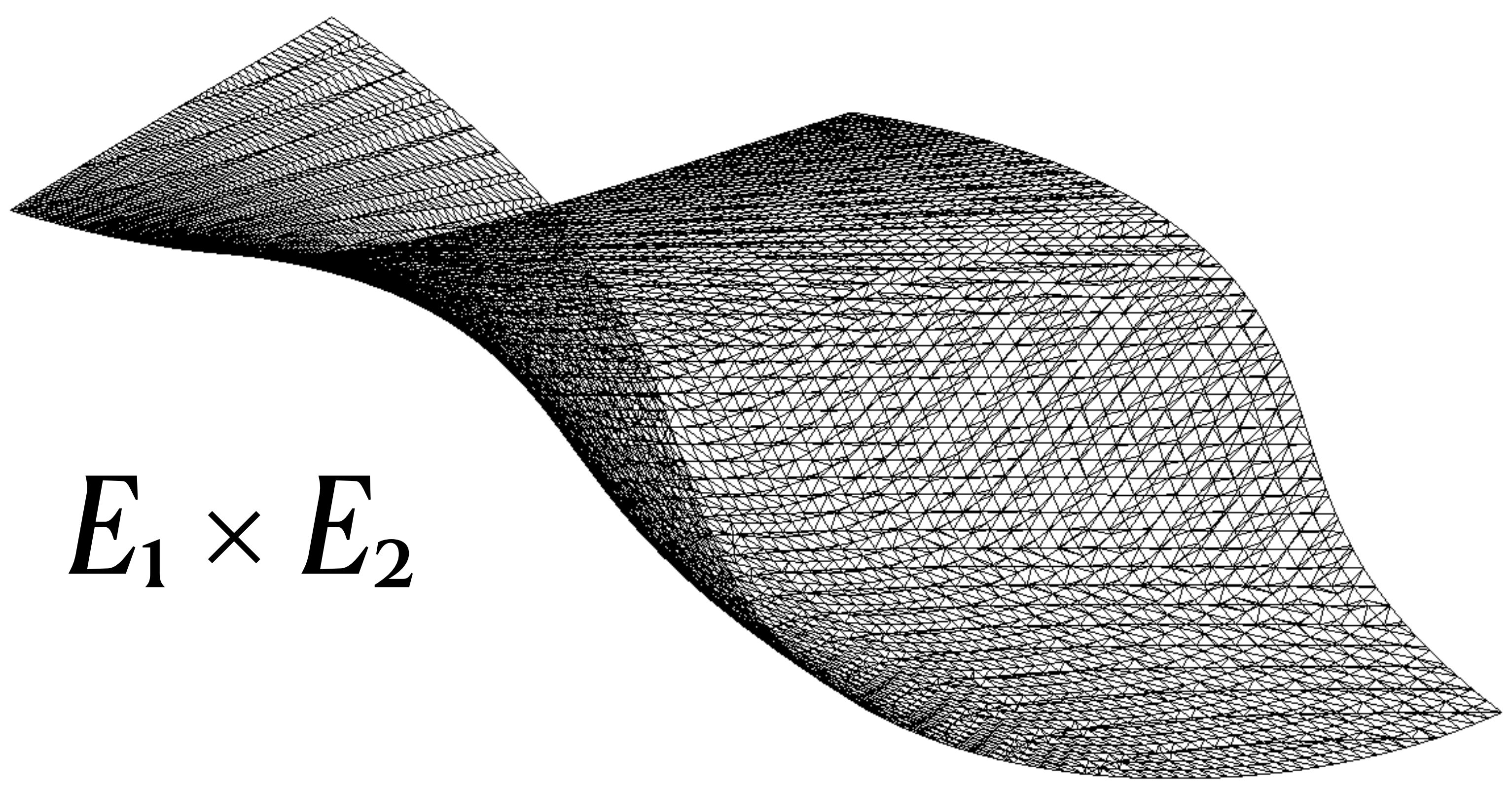
Abelian surface: surface that is also a group

- Example: product $E_1 \times E_2$





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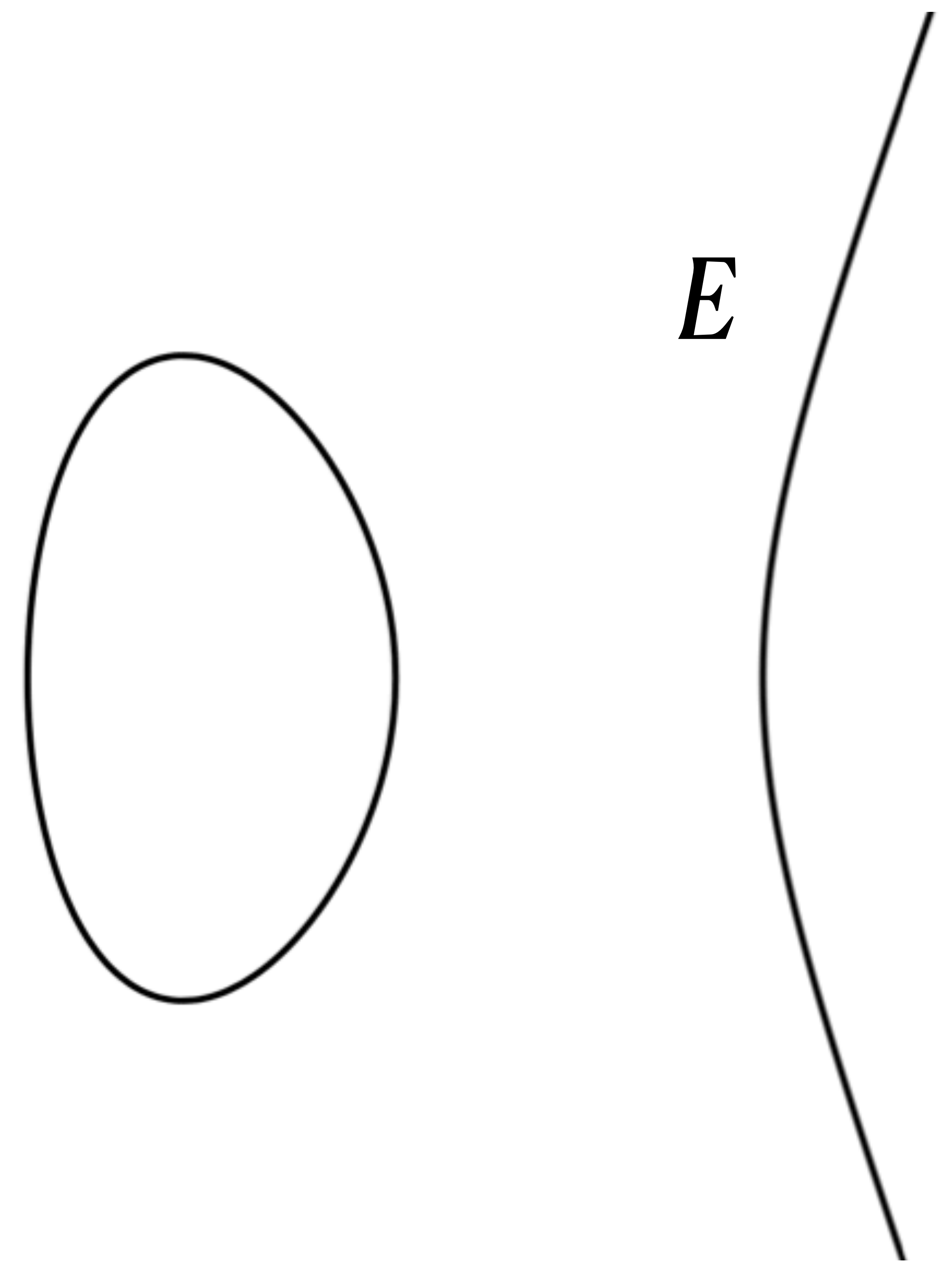
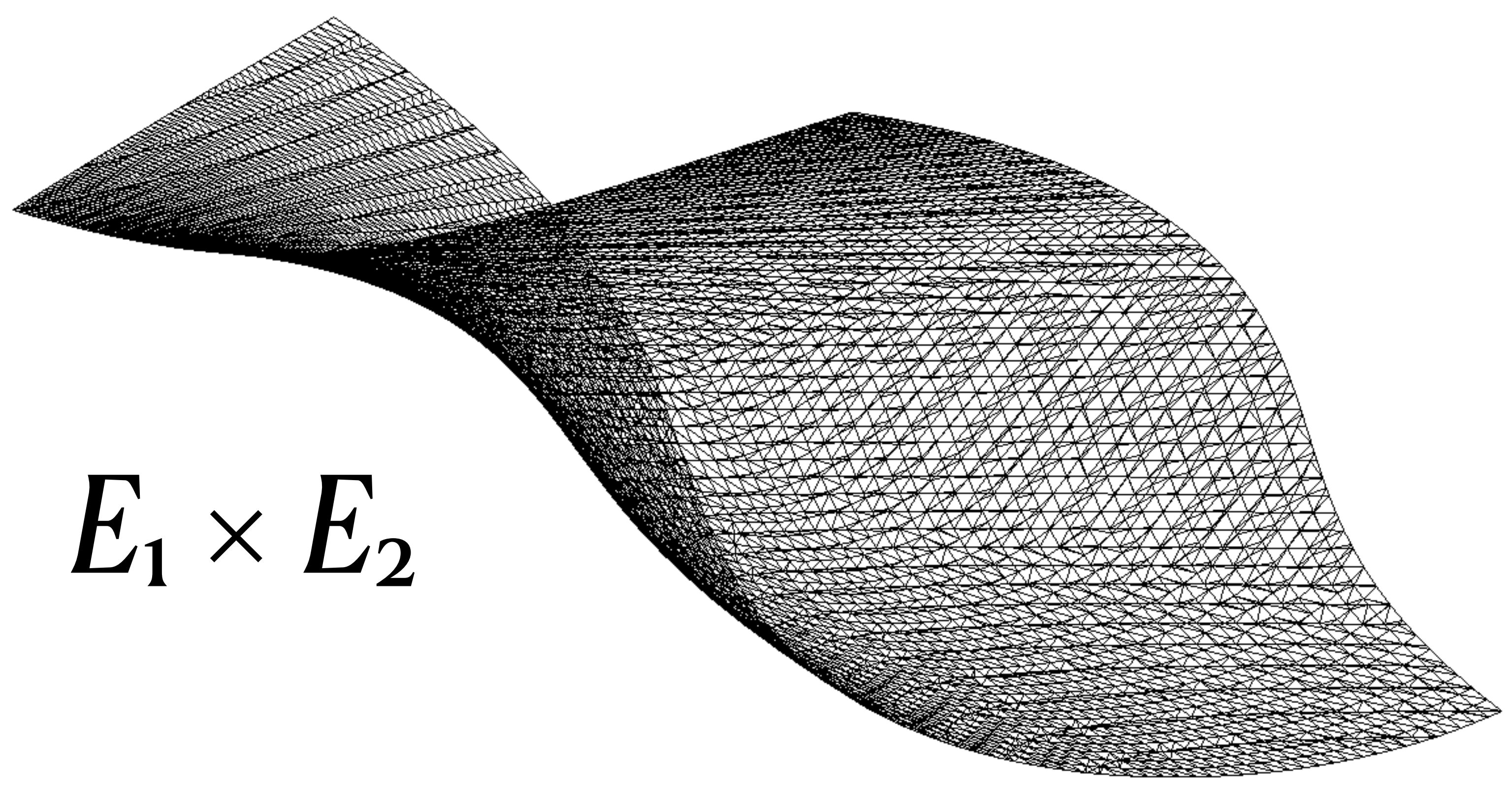
- Example: product $E_1 \times E_2 \times \ldots \times E_n$





Isogenies between products

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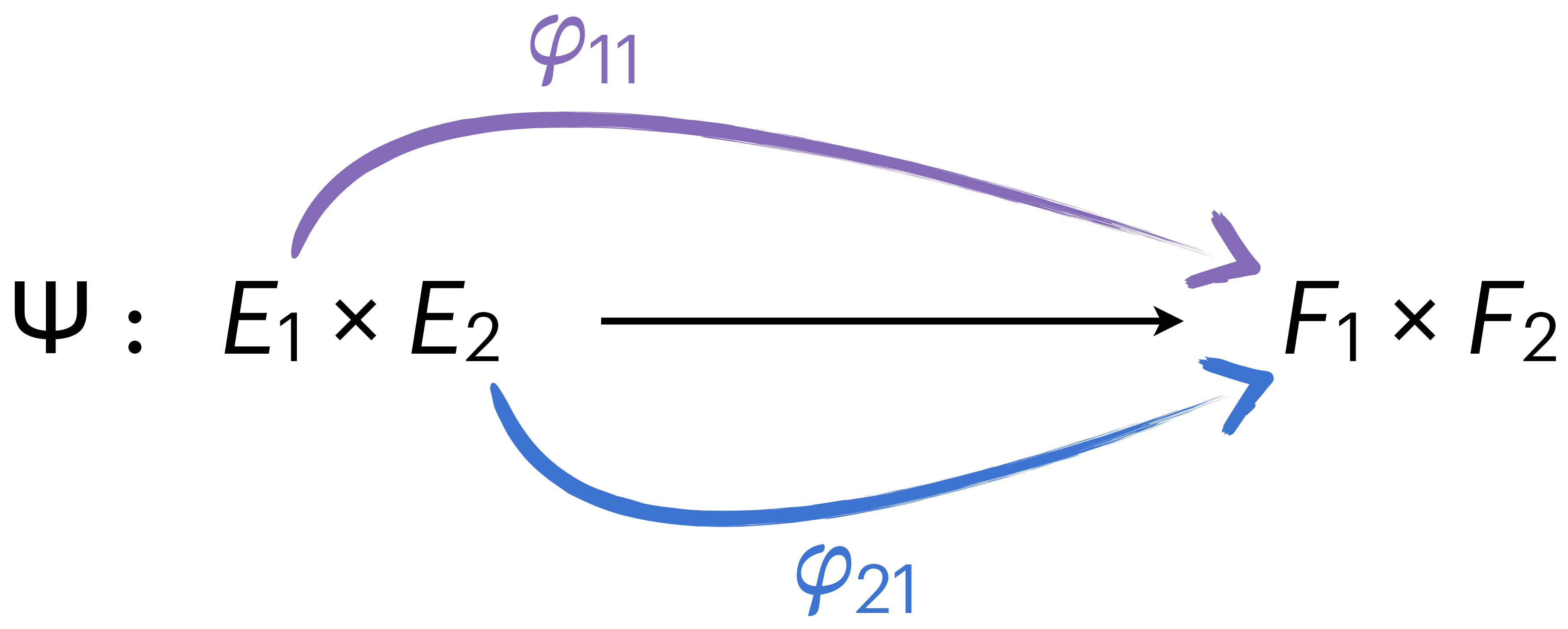

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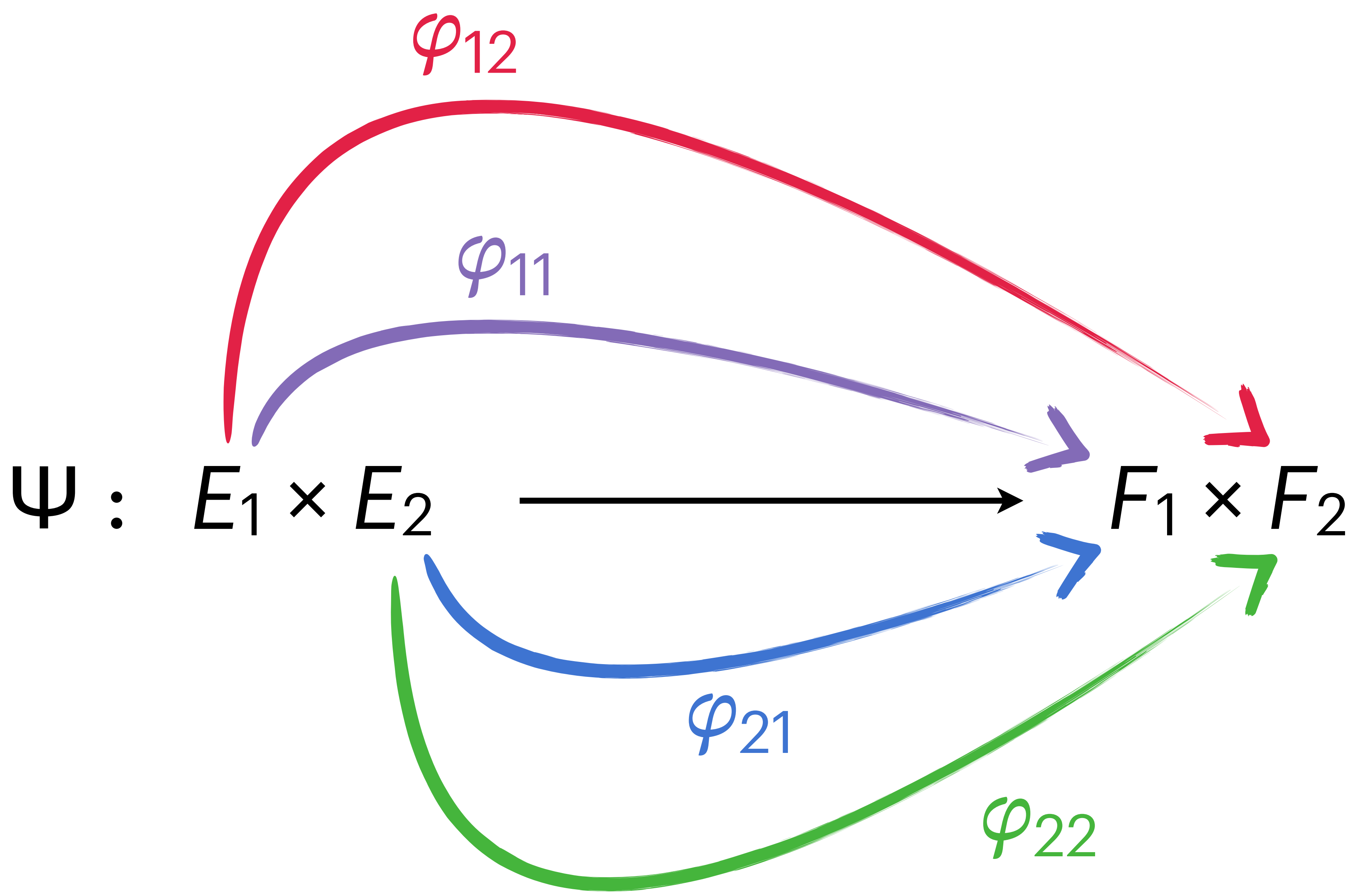

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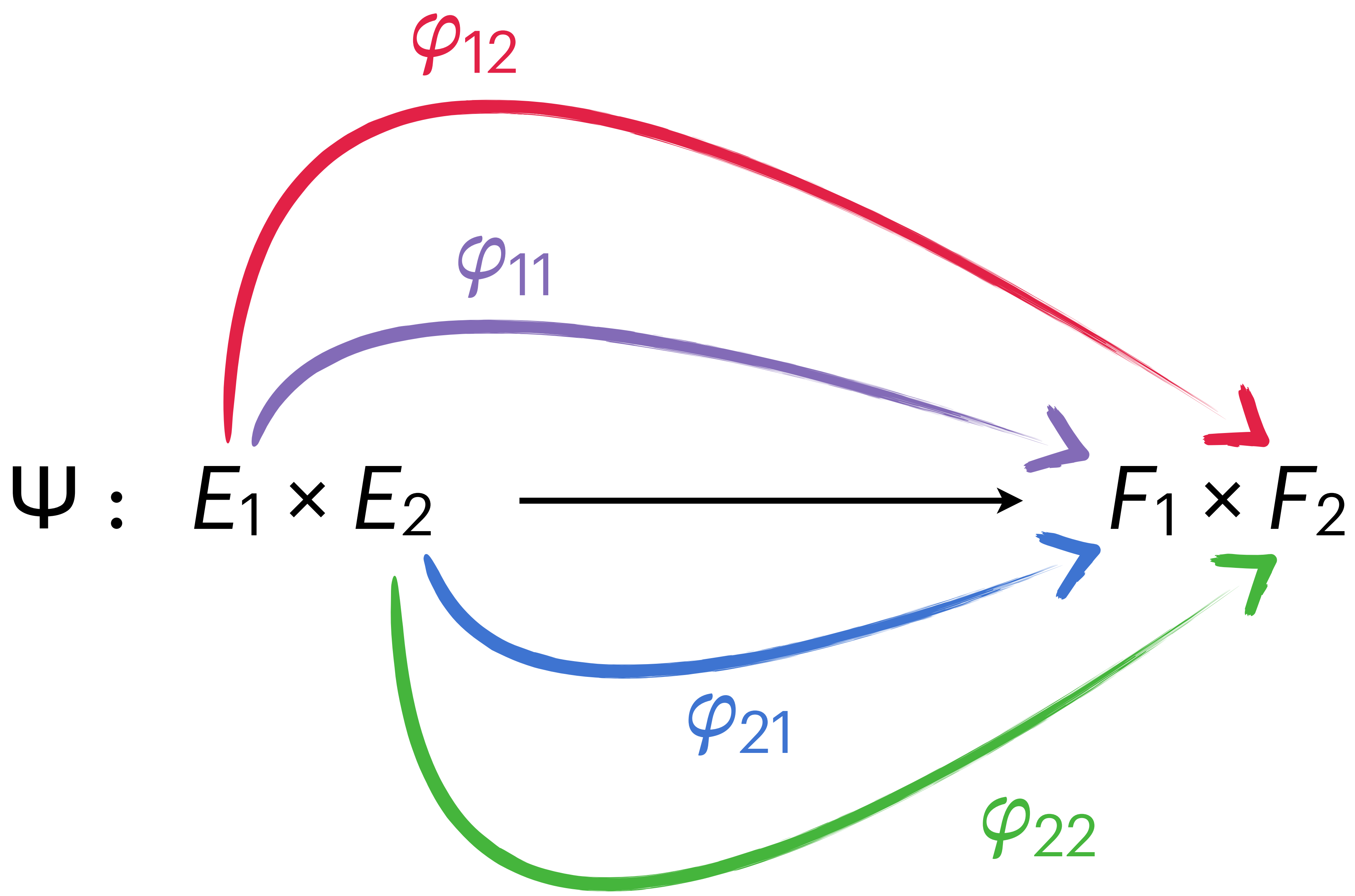

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- Given the kernel of a 2^n-isogeny, can evaluate it in polynomial time

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d & c & -b & a & 0 & & & -\hat{\varphi} \\
\varphi & & & & a & -b & -c & -d \\
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